

On the integrability of holomorphic vector fields

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Abstract

We determine topological and algebraic conditions for a germ of holomorphic foliation $\mathcal{F}(X)$ induced by a generic vector field X on $(\mathbb{C}^3, 0)$ to have a holomorphic first integral, i.e., a germ of holomorphic map $F: (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^2, 0)$ such that the leaves of $\mathcal{F}(X)$ are contained in the level curves of F .

1 Introduction

This paper is devoted to the relations between certain algebraic, topological and geometric properties of holomorphic foliations¹. The question of existence of first integrals for holomorphic foliations on $(\mathbb{C}^2, 0)$ depends heavily on the resolution algorithm due to Bendixson ([2]) and after rediscovered by Seidenberg ([13]). In dimension three the subject has been developed by Cano ([3]) and more recently, in the real analytic case, by Panazzolo ([12]), with the aid of the concept of weighted blow-up. This resolution leads to an ambient three-fold which is no more regular but has toroidal singularities (more precisely weighted projective spaces). On the other hand, as a consequence of a monomialization theorem due to Cutkosky ([6]), any singular holomorphic foliation on $(\mathbb{C}^3, 0)$ having a holomorphic first integral (see definition below) can be resolved with a finite number of monoidal transformations, avoiding the presence of singularities in the ambient three-fold of the resolved foliation. Generically, the singularities which appear after the resolution process are non-degenerate (and thus isolated), and have three (transverse) invariant planes through each singularity. Due to this fact, we consider along this work mainly singular holomorphic foliations satisfying the above properties.

In this work we study topological and algebraic conditions for the existence of holomorphic first integrals in a spirit similar to the classical approach for the two-dimensional case studied in [10]. Nevertheless, some differences arise with respect to the 2-dimensional case. The first is the fact that a stronger condition than the Siegel condition on the eigenvalues is required (see Definition 1 below).

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Also in order to find suitable integrable codimension one distributions containing the trajectories of X we need to apply some techniques from Partial Differential Equations (Problem 1 and §3.3).

1.1 Notation and statements

Denote the ring of germs of holomorphic functions on $(\mathbb{C}^n, 0)$ by \mathcal{O}_n and its maximal ideal by \mathcal{M}_n . We say that a germ of one-dimensional holomorphic foliation $\mathcal{F}(X)$ on $(\mathbb{C}^n, 0)$ given by the (germ of) holomorphic vector field $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ is *integrable*, or *has a holomorphic first integral*, if there is a germ of holomorphic map $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n-1}, 0)$ such that (a) the leaves of $\mathcal{F}(X)$ are contained in level curves of F or equivalently $i_X dF = 0$ and (b) F is a submersion off $\text{sing}(\mathcal{F}(X))$, the *singular set* of $\mathcal{F}(X)$. Further, let $f : (\mathbb{C}^n, 0) \rightarrow (\overline{\mathbb{C}}, \infty)$ be a meromorphic function, then we say that f is $\mathcal{F}(X)$ -*invariant* if $i_X df = 0$, i.e., if the leaves of $\mathcal{F}(X)$ are contained in the level sets of f . A germ of vector field on $(\mathbb{C}^n, 0)$ is *non-degenerate* if $DX(0)$ has just non-vanishing eigenvalues. Recall that generically $DX(0)$ has three distinct eigenvalues and thus is diagonalizable and X has an isolated singularity at the origin. Furthermore, from Poincaré-Dulac, Siegel and Brjuno linearization theorems and from [4], generically X leaves invariant the *coordinate hyperplanes* $x_1 \cdots x_n = 0$. Therefore we say that $\mathcal{F}(X)$ is *non-degenerate generic* if $DX(0)$ is diagonalizable and X leaves invariant the coordinate planes. For simplicity, we denote the set of germs of non-degenerate generic vector fields on $(\mathbb{C}^n, 0)$ by $\text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))$. Let $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))$, S a smooth integral curve of $\mathcal{F}(X)$ through the origin, and f a germ of $\mathcal{F}(X)$ -invariant meromorphic function. Then we denote by $\text{Hol}_\Sigma(\mathcal{F}(X), S)$ the holonomy of $\mathcal{F}(X)$ with respect to S evaluated at a section Σ transversal to S . Notice that $\Sigma \cap S = \{p\}$ is a single point, Σ is biholomorphic to a disc in \mathbb{C}^{n-1} with center corresponding to p and $\text{Hol}_\Sigma(\mathcal{F}(X), S)$ is conjugate to a subgroup of the group $\text{Diff}(\mathbb{C}^{n-1}, 0)$ of germs of complex diffeomorphisms fixing the origin in \mathbb{C}^{n-1} . Further we say that f is *adapted* to $(\mathcal{F}(X), S)$ if it can be written locally in the form $f = g/h$ where $g, h \in \mathcal{O}_n$ are relatively prime, $S \subset Z(g) \cap Z(h)$, where $Z(g)$ and $Z(h)$ denote the zero sets of g and h respectively, and $f|_\Sigma$ is pure meromorphic for a generic section Σ transversal to S . Denote the set of non-negative integers by \mathbb{N} and the set of positive integers by \mathbb{Z}_+ . Let $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ then we use respectively the following notation for monomials and their orders: $x^I := x_1^{i_1} \cdots x_n^{i_n}$, and $|I| = i_1 + \cdots + i_n$. Finally, we denote by C_n the germ of curve given by the union of the coordinate axes of $(\mathbb{C}^n, 0)$. Given a vector field X in $\text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ and a nonvanishing holomorphic function u in a neighborhood of the origin $0 \in \mathbb{C}^3$, the vector field $Y = uX$ also satisfies $Y \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$. We shall then say that X and Y are *tangent*. As

we will see (cf. Proposition 1), any integrable vector field $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfies the following condition:

Definition 1. Let $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$. We say that X *satisfies condition (\star)* if there is a real line $L \subset \mathbb{R}^2$ through the origin containing the eigenvalues of X such that one of the connected components $L \setminus \{0\}$ contains a single eigenvalue $\lambda(X)$ of X . In other words, not all the eigenvalues of X belong to the same connected component of $L \setminus \{0\}$. The above condition holds for X if and only if holds for any vector field Y such that X and Y are tangent. Condition (\star) implies that X is in the Siegel domain, but is stronger than this last. If X satisfies (\star) we denote by S_X the smooth invariant curve associate to $\lambda(X)$.

Our main result reads as follows:

Theorem 1. *Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfies condition (\star) and let S_X be the smooth invariant curve associate with the eigenvalue $\lambda(X)$. Then the following conditions are equivalent:*

- (i) *The leaves of $\mathcal{F}(X)$ are closed off $\text{sing}(\mathcal{F}(X))$;*
- (ii) *$\text{Hol}|_{\Sigma}(\mathcal{F}(X), S_X)$ has finite orbits;*
- (iii) *$\text{Hol}|_{\Sigma}(\mathcal{F}(X), S_X)$ is periodic (in particular linearizable and finite);*
- (iv) *$\mathcal{F}(X)$ has a holomorphic first integral.*

As a straightforward consequence we obtain the following topological criterion for the existence of $\mathcal{F}(X)$ -invariant meromorphic functions in $\text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$.

Theorem 2. *Let $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ and S_X be the invariant smooth curve associated with the eigenvalue $\lambda(X)$. Suppose that $\mathcal{F}(X)$ has closed leaves off $\text{sing}(\mathcal{F}(X))$, then there is an $\mathcal{F}(X)$ -invariant meromorphic function adapted to $(\mathcal{F}(X), S_X)$.*

2 Holomorphic first integrals and first jets

Here we study the necessary conditions on the eigenvalues of a vector field X in $\text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ in order to have X integrable. In particular we determine a criterion for linear vector fields to have holomorphic first integrals.

2.1 Algebraic criterion

Let us give an algebraic description of the first jets of integrable vector fields. First recall that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ has a holomorphic first integral $F = (f_1, f_2)$ if and only if $i_X df_j = 0$, $j = 1, 2$, and f_1, f_2 are transversal off the singular set of X . Let X be given in the system of coordinates (x_1, x_2, x_3) of $(\mathbb{C}^3, 0)$ by

$$X(x) = \lambda_1 x_1 (1 + a_1(x)) \frac{\partial}{\partial x_1} + \lambda_2 x_2 (1 + a_2(x)) \frac{\partial}{\partial x_2} + \lambda_3 x_3 (1 + a_3(x)) \frac{\partial}{\partial x_3}$$

where $a_1, a_2, a_3 \in \mathcal{M}_3$ and consider a holomorphic function $f \in \mathcal{M}_3$ preserving the coordinate axes. Then f must be given by $f(x) = \sum_{|N| \geq p} a_N x^N$ where $a_N \in \mathbb{C}$, $p \geq 2$ and $N \in \mathbb{N}^3 - C_3$. Since $\frac{\partial f}{\partial x_j} = \sum_{|N| \geq p} n_j a_N x^N x_j^{-1}$, then

$$\begin{aligned} J^p(df(X)) &= \frac{\partial f(x)}{\partial x_1} \cdot (\lambda_1 x_1) + \frac{\partial f(x)}{\partial x_2} \cdot (\lambda_2 x_2) + \frac{\partial f(x)}{\partial x_3} \cdot (\lambda_3 x_3) \\ &= \sum_{|N| \geq p} (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3) a_N x^N. \end{aligned}$$

Hence, if $i_X df = 0$ then

$$0 = (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3) a_N \text{ for all } |N| = p, N \in \mathbb{N}^3 - C_3. \quad (2.1)$$

From (2.1) it follows that $a_N = 0$ whenever $\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 \neq 0$, for all $N \in \mathbb{N}^3 - C_3$ such that $|N| = p$. In particular, in the absence of a resonance of the form

$$\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 = 0, \quad (2.2)$$

there will be no first integrals. Thus we have to study under what conditions on the eigenvalues of X we have a holomorphic first integral of the form $F = (f_1, f_2)$ where $\ker(df_1)$ is transverse to $\ker(df_2)$ off the singular set of X . For further purposes we prove the following technical result.

Lemma 1. *Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$, and let $(n_1, n_2, n_3), (m_1, m_2, m_3) \in \mathbb{N}^3 - C_3$ be linearly independent and satisfying (2.2) above. Then there are $m, n, k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}^*$ such that*

$$(\lambda_1, \lambda_2, \lambda_3) = \lambda(m, n, k)$$

and $m \cdot n \cdot k < 0$.

Proof. Since $(n_1, n_2, n_3), (m_1, m_2, m_3)$ are linearly independent then $(n_1 m_2 - n_2 m_1, n_1 m_3 - n_3 m_1, n_2 m_3 - n_3 m_2) \neq (0, 0, 0)$. We may suppose without loss of generality that $n_1 m_2 - n_2 m_1 \neq 0$. Now consider the system of equations

$$\begin{cases} n_1 \lambda_1 + n_2 \lambda_2 + n_3 \lambda_3 = 0 \\ m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3 = 0 \end{cases} \quad (2.3)$$

and sum the first equation multiplied by $-m_1$ with the second one multiplied by n_1 to obtain $(n_1m_2 - n_2m_1)\lambda_2 + (n_1m_3 - n_3m_1)\lambda_3 = 0$, i.e., $\lambda_2 = -\left(\frac{n_1m_3 - n_3m_1}{n_1m_2 - n_2m_1}\right)\lambda_3$.

Back to the first equation of (2.3) we have $\lambda_1 = \left(\frac{n_2m_3 - n_3m_2}{n_1m_2 - n_2m_1}\right)\lambda_3$. Hence $(\lambda_1, \lambda_2, \lambda_3) = \lambda(m, n, k)$ where $\lambda = \frac{\lambda_3}{n_1m_2 - n_2m_1}$, $m = n_2m_3 - n_3m_2$, $n = n_3m_1 - n_1m_3$ and $k = n_1m_2 - n_2m_1$. Thus

$$mnk = (m_1m_2m_3)^2\left(\frac{n_2}{m_2} - \frac{n_3}{m_3}\right)\left(\frac{n_3}{m_3} - \frac{n_1}{m_1}\right)\left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right).$$

An elementary evaluation shows that if two of the factors in the product $\left(\frac{n_2}{m_2} - \frac{n_3}{m_3}\right)\left(\frac{n_3}{m_3} - \frac{n_1}{m_1}\right)\left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)$ are positive, then the third one must be negative. \square

Proposition 1. *Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ has a holomorphic first integral, then $\mathcal{F}(X)$ can be given in local coordinates by a vector field of the form*

$$X(x) = mx_1(1 + a_1(x))\frac{\partial}{\partial x_1} + nx_2(1 + a_2(x))\frac{\partial}{\partial x_2} - kx_3(1 + a_3(x))\frac{\partial}{\partial x_3}$$

where $m, n, k \in \mathbb{Z}_+$ and $a_1, a_2, a_3 \in \mathcal{M}_3$. In particular X satisfies condition (\star) .

Proof. Suppose that $J^1(X) = \lambda_1x_1\frac{\partial}{\partial x_1} + \lambda_2x_2\frac{\partial}{\partial x_2} + \lambda_3x_3\frac{\partial}{\partial x_3}$, then Lemma 1 assures that its enough to prove that there is a pair of linearly independent vectors $M, N \in \mathbb{N}^3 - C_3$ satisfying (2.2). Suppose that $F = (f, g)$ is a first integral for X , with $f(x) = \sum_{|N| \geq p} a_N x^N$ and $g(x) = \sum_{|N| \geq q} b_N x^N$, i.e. with orders respectively p, q (≥ 2). From (2.2) we have that $0 = (\lambda_1n_1 + \lambda_2n_2 + \lambda_3n_3)a_N$ for all $|N| = p$. If there are two distinct $a_N, a_{N'} \neq 0$ then N and N' satisfy the desired condition. Reasoning in the same manner for g we just have to consider the case where $p = q$ and the terms of degree p of f and g are monomials, i.e., $f(x) = a_P x^P + \sum_{|N| \geq p+1} a_N x^N$ and $g(x) = b_P x^P + \sum_{|N| \geq p+1} b_N x^N$ with $|P| = p$, and $a_P, b_P \neq 0$. Now let $f_1 := \frac{1}{a_P}f - \frac{1}{b_P}g$, then it can be written in the form $f_1(x) = h_1(x^P) + \sum_{|N|=q, N \notin \langle P \rangle} c_N x^N + h.o.t$, where $h \in \mathcal{M}_1$ is a polynomial such that $\tau_1 := \deg(h_1) < q$, where $q = |N|$ is the less natural number such that there exists $c_N \neq 0$ form some $N \notin \langle P \rangle$ (notice that such q exists, since f and g are transversal off the origin). Pick inductively $f_k := f_{k-1} - h_{k-1}^{(\tau_{k-1})}(0) \left(\frac{1}{b_P}g\right)^{\tau_{k-1}}$, where $\tau_k := \deg(f_k)$, then after repeating this process a finite number of steps we have $k_0 \in \mathbb{Z}_+$ such that $f_{k_0}(x) = \sum_{|N|=q, N \notin \langle P \rangle} c_N x^N + h.o.t$. Since the set of $\mathcal{F}(X)$ -invariant holomorphic functions is a sub-ring of \mathcal{O}_3 , then f_{k_0} is an $\mathcal{F}(X)$ -invariant holomorphic function; in particular it satisfies (2.1). Therefore, it is enough to pick $R \notin \langle P \rangle$ such that $|R| = q$ and $c_R \neq 0$. \square

2.2 Linear vector fields

Let $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ be a linear vector field, then we determine under what conditions such a vector field has a holomorphic first integral. Recall from Proposition 1 that we just have to consider linear vector fields of the form $X(x) = mx_1 \frac{\partial}{\partial x_1} + nx_2 \frac{\partial}{\partial x_2} - kx_3 \frac{\partial}{\partial x_3}$ where $m, n, k \in \mathbb{N}$, and $m + n + k \geq 2$. Thus we have to study under what conditions on m, n, k we have a holomorphic first integral F for X . Write $F = (f_1, f_2)$ where $\ker(df_1)$ is transverse to $\ker(df_2)$ off the singular set of X . It is well-known that a germ of holomorphic foliation $\mathcal{F}(X)$ with $\text{codim}(\text{sing}(\mathcal{F}(X))) = 1$ can be extended to a foliation $\text{Sat}(\mathcal{F}(X))$ called the *saturated* of $\mathcal{F}(X)$ such that $\text{codim}(\text{sing}(\text{Sat}(\mathcal{F}))) \geq 2$.

Lemma 2. *Let $N, M \in \mathbb{N}^3 - C_3$ be two vectors satisfying (2.2), and let $f(x) = x^N$, $g(x) = x^M$. Then $\text{Sat}(df = 0)$ is transversal to $\text{Sat}(dg = 0)$ if and only if N and M are linearly independent.*

Proof. First suppose that the vectors are linearly dependent, then there is $k \in \mathbb{N}$ such that $m_j = kn_j$ for all $j = 1, 2, 3$. In particular $g = (f)^k$ and thus $dg = (f)^{k-1}df$. Therefore whenever $\text{Sat}(dg = 0)$ is non-singular it coincides with $\text{Sat}(df = 0)$. Conversely suppose that the vectors are linearly independent, then we have to show that $\text{Sat}(df = 0)$ is transverse to $\text{Sat}(dg = 0)$ away from the origin, or equivalently that $\text{Sat}(df \wedge dg)$ is non-singular off the origin. In fact,

$$\begin{aligned} df \wedge dg &= x_1^{n_1+m_1-1} x_2^{n_2+m_2-1} x_3^{n_3+m_3-1} [(n_1m_2 - n_2m_1)x_3 dx_1 \wedge dx_2 \\ &\quad + (n_1m_3 - n_3m_1)x_2 dx_1 \wedge dx_2 + (n_2m_3 - n_3m_2)x_1 dx_2 \wedge dx_3] \end{aligned}$$

Hence $\text{Sat}(df \wedge dg)$ is given by $\omega = 0$ where $\omega = (n_1m_2 - n_2m_1)x_3 dx_1 \wedge dx_2 + (n_1m_3 - n_3m_1)x_2 dx_1 \wedge dx_2 + (n_2m_3 - n_3m_2)x_1 dx_2 \wedge dx_3$ vanishes at the origin and whenever $n_1m_2 - n_2m_1 = n_1m_3 - n_3m_1 = n_2m_3 - n_3m_2 = 0$. But the later happens exactly when (n_1, n_2, n_3) and (m_1, m_2, m_3) are linearly dependent. \square

An algebraic characterization of integrable linear vector fields is given by the following result.

Lemma 3. *Any linear vector field of the form $X(x) = mx_1 \frac{\partial}{\partial x_1} + nx_2 \frac{\partial}{\partial x_2} - kx_3 \frac{\partial}{\partial x_3}$, where $(m, n, k) \in \mathbb{Z}_+^3$, has a holomorphic first integral of the form $F(x) = (x^N, x^M)$, where $N, M \in \mathbb{N}^3 - C_3$.*

Proof. From Lemma 2 and the calculations made in order to obtain (2.1), one can easily check that this is just a matter of finding two linearly independent solutions in $\mathbb{N}^3 - C_3$ for the homogeneous equation $mx + ny - kz = 0$. Therefore, we just have to pick $x_j := k\tilde{x}_j$ and $y_j := k\tilde{y}_j$, $j = 1, 2$, where $(\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2) \in \mathbb{N}^2$ are linearly independent. \square

2.3 Finite subgroups of germs of diffeomorphisms

Let $f: U \subset \mathbb{C}^n \rightarrow f(U) \subset \mathbb{C}^n$ be a homeomorphism in a neighborhood of the origin $0 \in \mathbb{C}^n$ with $f(0) = 0$. Given $x \in U$ denote by $\mathcal{O}_U(f, x)$ the f -orbit of x that does not leave U , i.e. $y \in \mathcal{O}_U(f, x)$ if and only if $\{x, f(x), \dots, y = f[k](x)\} \subset U$ or $\{x, f^{[-1]}(x), \dots, y = f^{[-k]}(x)\} \subset U$ for some $k \in \mathbb{N}$. The following result is very useful (cf. Theorem 3.1, [11]):

Theorem 3. *Let $F \in \text{Diff}(\mathbb{C}^2, 0)$. The group generated by F is finite if and only if there exists a neighborhood V of $0 \in \mathbb{C}^2$, such that:*

- i. $|\mathcal{O}_V(F, X)| < \infty$ for all $X \in V$ and
- ii. F leaves invariant infinitely many analytic varieties at 0 .

3 Holomorphic first integrals in $\text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$

In this section we describe a topological criterion for the integrability of $\text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ foliations.

3.1 The transverse geometry of \mathcal{F}

As a first step we study the existence of a transversely dicritical codimension 1 foliation \mathcal{G} tangent to the orbits of a vector field $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$.

Definition 2. Let \mathcal{F} be a germ of foliation by curves on $(\mathbb{C}^3, 0)$ given by the vector field X , then a germ of codimension one foliation \mathcal{G} on $(\mathbb{C}^3, 0)$ is said to be *tangent* to \mathcal{F} if its leaves contain the orbits of X .

Definition 3. Let \mathcal{G} be a codimension one foliation on $(\mathbb{C}^3, 0)$, and S a germ of curve through the origin, invariant by \mathcal{G} . We say that \mathcal{G} is *transversely dicritical* with respect to S if for any section Σ transversal to S in $(\mathbb{C}^3, 0)$ the restriction $\mathcal{G}|_\Sigma$ is dicritical.

Lemma 4. *Let \mathcal{F} be a germ of foliation by curves on $(\mathbb{C}^3, 0)$, S an invariant curve of \mathcal{F} through the origin and \mathcal{G} a codimension one foliation satisfying the following conditions:*

- (i) \mathcal{G} is tangent to X ;
- (ii) There is a section Σ transversal to S such that $\mathcal{G}|_\Sigma$ is dicritical.

Then \mathcal{G} is transversely dicritical with respect to S .

Proof. Since the orbits of X are contained in the leaves of \mathcal{G} , then they are invariant by the flow of X . Therefore if Σ' is another section transversal to S and $\phi : \Sigma \longrightarrow \Sigma'$ is an element of the holonomy pseudogroup of X with respect to S , then it is a diffeomorphism taking the leaves of $\mathcal{G}|_\Sigma$ onto the leaves of $\mathcal{G}|_{\Sigma'}$. \square

Our main concern here is the following: given a germ of foliation by curves \mathcal{F} induced by a germ of vector field of the form

$$X = mx(1 + a(x, y, z))\frac{\partial}{\partial x} + ny(1 + b(x, y, z))\frac{\partial}{\partial y} - kz(1 + c(x, y, z))\frac{\partial}{\partial z} \quad (3.4)$$

with $a, b, c \in \mathcal{M}_3$, find a codimension 1 germ of holomorphic foliation tangent to X which is transversely dicritical with respect to S . Thus, in order to achieve our task we have to find a solution of the following problem:

Problem 1. *Find a germ of holomorphic one form $\omega(x, y, z) = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ satisfying the following conditions:*

- (i) (*integrability*) $\omega \wedge d\omega = 0$;
- (ii) (*tangency*) $i_X \omega = 0$;
- (iii) (*dicriticalness*) $(\mathcal{G} : \omega = 0)$ is transversely dicritical with respect to $S := (z = 0)$.

3.2 Perturbations of some dicritical foliations on $(\mathbb{C}^2, 0)$

In this section we shall deal with the question of transversal dicriticality of a codimension one foliation $\mathcal{G} : (\omega = 0)$ with respect to S . We begin with a vector field $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ of the form $X = mx(1 + P(x, y, z))\frac{\partial}{\partial x} + ny(1 + Q(x, y, z))\frac{\partial}{\partial y} - kz(1 + R(x, y, z))\frac{\partial}{\partial z}$ where $m, n, k \in \mathbb{N}$ and $P, Q, R \in \mathcal{M}_3$ we can define the one-parameter family of 1-forms

$$\omega_z(x, y) := \omega(x, y, z) = mx(1 + P_z(x, y))dy - ny(1 + Q_z(x, y))dx \quad (3.5)$$

where $\omega_z \in \Lambda^1(\mathbb{C}^2, 0)$, $P_z(x, y) := P_z(x, y, z)$ and $Q_z(x, y) := Q_z(x, y, z)$, for each fixed $z \in (\mathbb{C}, 0)$. We denote by $\mathcal{F}(X)'_z$ the one-parameter family of germs of foliations defined on $(\mathbb{C}^2, 0)$ by ω_z .

Remark 1. Notice that for X as above we have $\lambda(X) = -k$ and S_X is the z -axis. Also the kernel of ω denoted $\text{Ker}(\omega)$ is a holomorphic distribution on $(\mathbb{C}^3, 0)$ tangent to the orbits of X , since $i_X \omega = 0$. Moreover, if $\Sigma_z := (z = \text{const.})$ and $i_z : \Sigma_z \longrightarrow \mathbb{C}^3$ is the inclusion map, then $(i_z)^* X$ is tangent to $(i_z)^* \omega$.

Keeping in mind this framework, we consider the following situation. Let ω_z be an one-parameter holomorphic family of 1-forms on $(\mathbb{C}^2, 0)$ given by

$$\omega_z(x, y) := mx(1 + P_z(x, y))dy - ny(1 + Q_z(x, y))dx$$

where $P, Q \in \mathcal{M}_3$, $P_z(x, y) := P_z(x, y, z)$ and $Q_z(x, y) := Q(x, y, z)$, for each fixed $z \in (\mathbb{C}, 0)$. We shall denote by \mathcal{F}_z the one-parameter family of germs of foliations defined on $(\mathbb{C}^2, 0)$ by ω_z .

Lemma 5. *For each neighborhood U of the origin in \mathbb{C} , there is $z \in U$ such that \mathcal{F}_z is dicritical.*

Proof. First notice that \mathcal{F}_z may be given by $\omega_z(x, y) := x(1 + xA + yB)dy - y(\frac{p}{q} + g(z) + xC + yD)dx$, where p and q are relatively prime positive integers, $A, B, C, D \in \mathcal{O}_3$, and $g \in \mathcal{M}_1$. In particular, we may suppose that $|g(z)| < \varepsilon$ for any z sufficiently close to the origin. Moreover, since g is an open map, we can find a sequence $z_n \rightarrow 0$ such that $g(z_n) = \frac{1}{q^n}$. Therefore, we just need to verify that $(\mathcal{F}_{z_n} : \omega_{z_n} = 0)$ is dicritical; but this comes immediately from Lemma 6 below. \square

Lemma 6. *Let $\mathcal{F}(\omega)$ be a foliation in $(\mathbb{C}^2, 0)$ be given by a one-form $\omega(x, y) = mx(1 + xA(x, y) + yB(x, y))dy - ny(1 + xC(x, y) + yD(x, y))dx$ where $A, B, C, D \in \mathcal{O}_2$. Then $\mathcal{F}(\omega)$ has the same resolution tree of $\mathcal{F}(\omega_{m,n})$ where $\omega_{m,n}(x, y) = mxdy - nydx$. In particular $\mathcal{F}(\omega)$ is dicritical.*

Proof. After one blow-up we obtain

$$\tilde{\omega}_0(x, t) = \frac{1}{x}\pi^*\omega(x, t) = x(m + x\tilde{A} + tx^2\tilde{B})dt + t((m - n) + x(\tilde{A} - \tilde{C}) + tx(\tilde{B} - \tilde{D}))dx$$

and

$$\tilde{\omega}_\infty(u, y) = \frac{1}{y}\pi^*\omega(u, y) = u((m - n) + uy(\tilde{A} - \tilde{C}) + y(\tilde{B} - \tilde{D}))dy - y(n + uy\tilde{C} + y\tilde{D})dt +$$

where $\tilde{A} = \pi^*A, \tilde{B} = \pi^*B, \tilde{C} = \pi^*C$ and $\tilde{D} = \pi^*D$. Without loss of generality we may suppose that $m > n$. In this case we have a reduced singularity at $(t, x) = 0$ and a non-reduced one at $(u, y) = (0, 0)$, with the same character of the original one. Therefore, Euclid's algorithm assures that after a finite number of blow-ups we obtain a linear chain of projective lines, any of them with two singularities, and just one of the singularities in the exceptional divisor being non-reduced: the one given in local coordinates (\tilde{x}, \tilde{y}) by $\omega_{m,n}(\tilde{x}, \tilde{y}) = \tilde{x}(1 + \tilde{x}A_{m,n}(\tilde{x}, \tilde{y}) + \tilde{y}B_{m,n}(\tilde{x}, \tilde{y}))d\tilde{y} - \tilde{y}(1 + \tilde{x}C_{m,n}(\tilde{x}, \tilde{y}) + \tilde{y}D_{m,n}(\tilde{x}, \tilde{y}))d\tilde{x}$. Hence (3.6) and (3.6) show that one further blow-up leads to a foliation transversal to the new projective line appearing in the divisor; this is exactly the resolution tree of $\omega_{m,n}$. \square

3.3 Solution to Problem 1

We verify that we can find (local) solutions for the Problem 1.

3.3.1 Tangency and dicriticality conditions

First notice that Lemmas 4 and 5 suggest us to search ω of the form $\omega = ny(1+q(x, y, z))dx - mx(1+p(x, y, z))dy + R(x, y, z)dz$. Thus, from the tangency condition of Problem 1, we can determine R in terms of p and q as follows. From (3.4) we shall have $0 = i_X\omega = mnxy[a - b + q - p + aq - bp] - kz(1+c)R$. If we set $K := a - b + q - p + aq - bp$ we shall have

$$0 = mnxyK - kz(1+c)R \quad (3.6)$$

Recall that our main goal here is to obtain R as an explicit holomorphic function in terms of the coordinates of X and the first two coordinates of ω . Therefore, we shall need K to be a multiple of z . But notice that, in the particular case where $C \equiv 0$ we just have to take $p = a$ and $q = b$. This suggests us to take $q := b + z\tilde{q}$ and $p := a + z\tilde{p}$, for some $\tilde{p}, \tilde{q} \in \mathcal{O}_3$. Under this assumption we shall have after some simple calculations that $K = z[(a+1)\tilde{q} - (b+1)\tilde{p}]$. Therefore (3.6) turns out to be $0 = mnxyz[(a+1)\tilde{q} - (b+1)\tilde{p}] - kz(1+c)R$. Thus

$$R = \frac{mnxy[(a+1)\tilde{q} - (b+1)\tilde{p}]}{k(1+c)}. \quad (3.7)$$

3.3.2 Integrability condition

First notice that for $\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ the integrability condition $\omega \wedge d\omega = 0$ is equivalent to the first order PDE

$$(-P_y + Q_x)R - (-P_z + R_x)Q + (-Q_z + R_y)P = 0, \quad (3.8)$$

therefore we have to apply (3.7) in (3.8) in order to study the solutions of Problem 1. For simplicity, let us first introduce the following notation: $\bar{a} := 1 + a, \bar{b} := 1 + b, \bar{c} := 1 + c, \tilde{q} := \overline{c\tilde{q}}$ and $\tilde{p} := \overline{c\tilde{p}}$. Then, if we take P, Q and R in the form

$$\begin{cases} P = ny(\bar{b} + z\overline{c\tilde{q}}), \\ Q = -mx(\bar{a} + z\overline{c\tilde{p}}), \\ R = \frac{mn}{k}xy(\overline{a\tilde{q}} - \overline{b\tilde{p}}), \end{cases} \quad (3.9)$$

we shall obtain after some lengthy calculations that

$$\begin{aligned}
0 = & mx[(\bar{a}^2\bar{q}_x - \bar{a}\bar{b}_x\bar{p} - \bar{a}\bar{b}_x\bar{p} + z\bar{a}_x\bar{c}\bar{p}\bar{q} + z\bar{a}\bar{c}\bar{p}\bar{q}_x - z\bar{b}_x\bar{c}\bar{p}^2) \\
& - (z\bar{a}\bar{c}_x\bar{p}\bar{q} + z\bar{a}\bar{c}\bar{p}_x\bar{q} - \bar{a}_x\bar{b}\bar{p} - z\bar{b}\bar{c}_x\bar{p}^2)] \\
& + ny[(\bar{b}\bar{a}_y\bar{q} + \bar{b}\bar{a}_y\bar{q} - \bar{b}^2\bar{p}_y + z\bar{a}_y\bar{c}\bar{q}^2 - z\bar{b}_y\bar{c}\bar{p}\bar{q} - z\bar{b}\bar{c}\bar{q}\bar{p}_y) \\
& - (\bar{a}\bar{b}_y\bar{q} + z\bar{a}\bar{c}_y\bar{q}^2 - z\bar{b}\bar{c}_y\bar{p}\bar{q} - z\bar{b}\bar{c}\bar{p}\bar{q}_y)] \\
& + k[(\bar{a}_z\bar{b} + \bar{b}\bar{c}\bar{p} + z\bar{b}\bar{c}_z\bar{p} + z\bar{b}\bar{c}\bar{p}_z + z\bar{a}_z\bar{c}\bar{q} + z^2\bar{c}^2\bar{p}_z\bar{q}) \\
& - (\bar{a}\bar{b}_z + \bar{a}\bar{c}\bar{q} + z\bar{a}\bar{c}_z\bar{q} + z\bar{a}\bar{c}\bar{q}_z + z\bar{b}_z\bar{c}\bar{p} + z^2\bar{c}^2\bar{p}\bar{q}_z)]
\end{aligned}$$

Since we are interested in the foliation defined by ω (and not in ω itself) and since \bar{b} is a unity, then we may suppose without loss of generality that $\bar{q} = 0$ (In fact, from (3.9) we just have to divide ω by $(1 + z\bar{c}\bar{q}/\bar{b})$). Analogously, we are not interested in X itself but in the foliation \mathcal{F} defined by it. Thus we may also suppose without loss of generality that $\bar{b} = 1$ (just divide X by $\bar{b} = 1 + b$). Under these assumptions the above equation turns out to be

$$0 = ka_z + (mx\bar{a}_x + k\bar{c} + kz\bar{c}_z)\bar{p} + mxz\bar{c}_x\bar{p}^2 - mxa\bar{p}_x - ny\bar{p}_y + kz\bar{c}\bar{p}_z \quad (3.10)$$

Now notice that a germ of holomorphic function defined on any of the coordinate planes can be taken as a Cauchy condition for the first order PDE (3.10). Since the origin is a non-characteristic point for (3.10), then we may apply the classical method of characteristics for non-linear PDEs in order to find a (local) solution to (3.10) with the above initial condition (cf. e.g. [9]). This solves Problem 1, as desired.

3.4 The existence of a topological criterion

Here we prove Theorem 1. For this sake let us first recall some facts proved along this article and introduce some terminology. Let $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfying condition (\star) in § 1.1. We can assume that the curve S_X is the z -axis so that we can apply the results in § 3.2. Let $\Sigma_z := (z = \text{const.})$ be a section transversal to S_X and $\text{Hol}_{\Sigma_z}(\mathcal{F}(X), S_X)$ the holonomy of $\mathcal{F}(X)$ with respect to S_X , evaluated at Σ_z .

Proof of Theorem 1. First let us prove that (i) implies (ii). Suppose all the leaves of $\mathcal{F}(X)$ are closed off $\text{sing}(\mathcal{F}(X))$. Given a leaf L of \mathcal{F} it follows that the closure $\bar{L} \subset L \cup \text{sing}(\mathcal{F})$ is an analytic subset of pure dimension one in \mathbb{C}^3 and since this leaf is transversal to Σ_z , we conclude that $\bar{L} \cap \Sigma_z$ is a finite set. On the other hand, given a point $x \in L \cap \Sigma_z$ its orbit in the holonomy group is also contained in $L \cap \Sigma_z$ so that it is a finite set. Thus the orbits of H_z are finite proving (ii).

Now let us verify that (ii) implies (iii). First recall from Lemma 5 that there is $z_0 \in (\mathbb{C}, 0)$ may be choose conveniently in such a way that $\mathcal{F}(X)'_{z_0}$ is dicritical. Now consider a simple loop γ around the origin, inside the z -axis, starting from z_0 . Pick a leaf L of $\mathcal{F}(X)'_{z_0}$ and consider the liftings of γ starting at points of L , along the trajectories of $\mathcal{F}(X)$. Then these liftings form a three dimensional real manifold, say S_L , whose intersection with Σ_{z_0} is given by L and L' (see Figure 1). In particular if $h := h_\gamma$ is the generator of $\text{Hol}_{\Sigma_z}(\mathcal{F}(X), S_X)$, then $L' = h(L)$. If ω is a solution to Problem 1, then S_L is tangent to $\text{Ker}(\omega)$, and $S_L \cap \Sigma_{z_0}$ is tangent to the distribution $\text{Ker}(\omega_{z_0})$. But the last one is integrable and given by $\mathcal{F}(X)'_{z_0}$, then L' is a leaf of $\mathcal{F}(X)'_{z_0}$. Thus, since $\mathcal{F}(X)'_{z_0}$ is dicritical and the orbits of h are finite, then h preserves an infinite number of separatrices in Σ_{z_0} . Therefore, Theorem 3 assures that h is periodic (in particular linearizable and finite). Now let us prove that (iii) implies (iv). From [7] this comes immediately from Proposition 1 and the fact that $\mathcal{F}(X) \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$, since the holonomy of $\mathcal{F}(X)$ is linearizable. Therefore one may suppose without loss of generality that $X(x) = mx_1 \frac{\partial}{\partial x_1} + nx_2 \frac{\partial}{\partial x_2} - kx_3 \frac{\partial}{\partial x_3}$ and the result follows from Lemma 3. Finally, since analytic varieties are closed, then (i) comes straightforward from (iv). \square

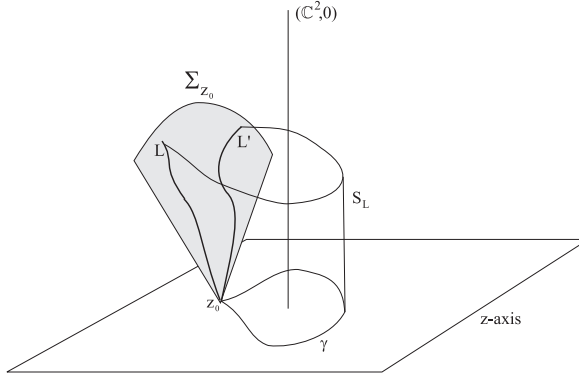


Figure 1. The liftings of γ along the leaves of $\mathcal{F}(X)$ starting at points of L .

Proof of Theorem 2. Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ has all its leaves closed off the singular set, then by Theorem 1 we may assume without loss of generality that $X(x) = mx_1 \frac{\partial}{\partial x_1} + nx_2 \frac{\partial}{\partial x_2} - kx_3 \frac{\partial}{\partial x_3}$ and has a holomorphic first integral of the form $F(x_1, x_2, x_3) = (x^P, x^Q)$ for some linearly independent $P, Q \in \mathbb{N}^3 - C_3$ with $|P|, |Q| \geq 2$. In particular we have

$$\begin{cases} p_1 m + p_2 n - p_3 k = 0 \\ q_1 m + q_2 n - q_3 k = 0 \end{cases}$$

and thus $(p_1q_3 - p_3q_1)m + (p_2q_3 - p_3q_2)n = 0$. Since $m, n > 0$ and P, Q are linearly independent, then $(p_1q_3 - p_3q_1)(p_2q_3 - p_3q_2) < 0$. Therefore $f(x) := x^{q_3 \cdot P} / x^{-p_3 \cdot Q}$ is an $\mathcal{F}(X)$ -invariant meromorphic function adapted to $(\mathcal{F}(X), S_X)$. \square

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